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TEXTBOOK ANALYSIS OF COVARIANCE

-- IS IT CORRECT?

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Technical Report #204

May, 1981

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The preparation of this paper was partially supported by the Office of Naval Research Contract N00014-80-C-0637 at Carnegie-Mellon University. Reproduction in whole or in part is permitted for any purpose of the United States Government.

Summary

For unbalanced Model I ANOVA data there is a controversy in the literature -- should the partial SS's be computed hierarchically or should they be computed after fitting all other model terms including higher-order interactions? A similar problem exists in the Model I analysis of covariance (ANCOVA), yet the usual prescription given in textbooks uses only the non-hierarchical approach. Using two classic examples of ANCOVA, this paper examines the two approaches to ANCOVA calculations.

1. Introduction

In recent years much attention has been given to the question as to what is the correct method of analysis for data from an unbalanced Model I analysis of variance (ANOVA). A principal area of contention has been whether the SS to be used in the numerator of an F-test should be computed assuming higher-order interactions are zero, or whether it should be the partial sum of squares for the effect under test after fitting all other model terms including higher order interactions. Both approaches have their advocates. Of course, in the balanced ANOVA case there is no dispute, since the two approaches are equivalent there.

What does not seem to have been recognized, or at least clearly exposited in the literature and especially in textbook discussions, is that a question exists as to what is the correct method of analysis in the Model I analysis of covariance (ANCOVA), even in the case of a balanced design. Unless the covariate is itself "secretly balanced", its inclusion in the analysis automatically unbalances a balanced design. The choice of an appropriate method of analysis is (or should be) essentially the same for both the unbalanced Model I ANOVA and the Model I ANCOVA. How is the matter dealt with in textbooks?

The usual prescription given in textbooks for computing an adjusted sum of squares for one out of several sources of variation follows the classic prescription of Cochran (1957), reproduced here in Table 1, and yields:

$$SS_{adj} = [E_{yy}+T_{yy}-(E_{xy}+T_{xy})^2/(E_{xx}+T_{xx})]-[E_{yy}-E_{xy}^2/E_{xx}]$$

$$= T_{yy} + E_{xy}^2/E_{xx} - (E_{xy}+T_{xy})^2/(E_{xx}+T_{xx}). \qquad (1)$$

The first form of SS_{adj} is in the form of a difference of two residual SS, one excluding the effect, the other including it. It reveals that SS_{adj} is in fact a partial SS for the effect, after fitting all of the other terms in the unadjusted ANOVA <u>including</u> the corresponding interaction terms, i.e., the second approach mentioned above. In this paper we refer to an analysis based on such a prescription as the "textbook" analysis.

-- Table 1 goes about here --

If we were to write out the ANCOVA model for a design with at least two dimensions (sources of variation) as a regression model, omitting design variables corresponding to interaction terms, and then compute the SS for a specific source of variation, we would typically get a different answer -- one which corresponds to assuming that the higher-order interaction terms are zero, i.e., the first approach mentioned above. Here, we shall refer to such a SS calculation as being "hierarchical", as contrasted with the non-hierarchical nature of the "textbook" analysis. The hierarchical ANCOVA approach is notable for its absence from the major textbooks dealing with experimental design. However, in the balanced case it is no more computationally complex than the textbook ANCOVA and in the unbalanced case may in fact be simpler. The adjusted SS for a main effect (effect lower in hierarchy than interactions) is

$$SS_{adj} = [E_{yy}+I_{yy}+T_{yy} - \frac{(E_{xy}+I_{xy}+T_{xy})^2}{(E_{xx}+I_{xx}+T_{xx})}] - [E_{yy}+I_{yy} - \frac{(E_{xy}+I_{xy})^2}{(E_{xx}+I_{xx})}]$$

$$= T_{yy} + \frac{(E_{xy}+I_{xy})^2}{(E_{xx}+I_{xx})} - \frac{(E_{xy}+I_{xy}+T_{xy})^2}{(E_{xx}+I_{xx}+T_{xx})} \qquad (2)$$

where I_{yy}, I_{xy}, I_{xx} are sums of interaction SS's (terms higher in hierarchy than T_{yy}, etc.)

In Section 2 we illustrate the two solutions to the ANCOVA calculations using two classic examples, one taken from the 2nd edition of the text by Cochran and Cox (1957), and the other (an unbalanced design) from Federer (1957), in the special Biometrics issue on ANCOVA. In Section 3 we comment on these and other textbook examples we have examined, make some related points regarding assumptions for ANCOVA calculations, and argue that those who oppose the non-hierarchical approach to unbalanced ANOVA, should also eschew it in the ANCOVA.

2. Two Examples

We begin with an example from Federer (1957), the data for which are included here in Table 2. The structure here is actually a one-way design with 5 treatments (levels) in randomized blocks with unequal replication within blocks. Federer's analysis treats this as a two-way design, keeping the block × treatment interaction separate from the within block error and implicitly assuming the unweighted sums of the block effects and of the treatment effects are zero in defining interactions. Federer does not, however, test for blocks, and thus recognizes their special role. Our analysis parallels Federer's in both these regards.

-- Table 2 goes about here --

Federer's analysis is notable because he includes all of the SS's necessary to do both the textbook and hierarchical analyses in his Table 10, but he deliberately chooses to use the "textbook" analysis, testing treatment effects after eliminating blocks, <u>interactions</u>, and the covariate. The left-hand panel of Table 3 contains the relevant SS's from Federer (slightly corrected), and the middle panel contains his "text-book" analysis of treatment effects. His calculations yield

$$\begin{aligned} SS_{adj}(T) &= SS(T|B, T\times B, x) \\ &= 1606.67 + \frac{(6.00)^2}{4.00} - \frac{(367.33+6.00)^2}{(172.53+4.00)} \\ &= 826.14. \end{aligned}$$

The right-hand panel of Table 3 contains the "hierarchical" analysis for which

$$SS_{adj}(T) = SS(T|B,x)$$

$$= 1320.96 + \frac{(139.80+6.00)^2}{(89.47+4.00)}$$

$$- \frac{(303.81+139.80+6.00)^2}{(136.12+89.47+4.00)}$$

$$= 667.91.$$

(As we would expect the line for the interaction term is the same in both analyses.) The F-ratio for treatments in the "textbook" approach is 3.60 which falls just below $F_{.10}(4,4) = 4.11$, while the F value in the "hierarchical" analysis, 2.91, is closer to $F_{.25}(4,4) = 2.06$, than to the 0.10 value. Here, the two solutions differ sufficiently to make a difference in the conclusions that might be drawn.

-- Table 3 goes about here --

Our second example is the primary illustration of ANCOVA in Cochran and Cox (1957). The data are given there in Table 3.1, p. 46, and come from an experiment in 4 blocks of 12 plots each. Four fumigants were used at dosage levels 0, 1, and 2 in each block, and thus in each block we have 4 controls, 1 plot of each fumigant at level 1, and 1 plot of each at level 2. The model used in Cochran and Cox's ANOVA analysis in Table 3.5 can be written as

$$y_{ijk} = \mu + B_k + \alpha_1(j-1) + \alpha_2(j-1)^2 + \beta_{1i}(j-1) + \beta_{2i}(j-1)^2 + \epsilon_{ijk} \qquad (3)$$

where

B_k is the block effect, i indexes the fumigants, j indexes level, and

$$\beta_{\ell 1} + \beta_{\ell 2} + \beta_{\ell 3} + \beta_{\ell 4} = 0 \tag{4}$$

for $\ell = 1,2$. The response y is the number of eelworm cysts in a sample of 400 grams of soil. The covariate used by Cochran and Cox in their analysis of covariance on pp. 84-86 is the count for a sample of 400 grams of soil before fumigation.

What makes this example of some interest is that Cochran and Cox use orthogonal polynomial contrasts to partition the treatment SS appropriately for the hypotheses of interest. These are implicitly determined in a particular order in which (j-1) is made orthogonal to the constant and $(j-1)^2$ is made orthogonal to the constant and (j-1). Given these contrasts there is no dispute as to how one should compute the ANOVA table corresponding to the model in expression (3). Once we add the covariate the model becomes

$$y_{ijk} = \mu + B_k + \alpha_1(j-1) + \alpha_2(j-1)^2 + \beta_{1i}(j-1) \\ + \beta_{2i}(j-1)^2 + \gamma(x_{ijk} - \bar{x}) + \varepsilon_{ijk} \tag{5}$$

and one would hope that the ANCOVA adjusted SS's retain the orthogonality of the original analysis. Sadly, the introduction of the covariate destroys the orthogonality.

In the left-hand panel of Table 4 we give the basic sum of squares and products from Cochran and Cox's Table 3.9;

in the middle panel we give their "textbook" analysis values of SS_{adj} and F for each line. Finally, in the right-hand panel we give the hierarchical analysis.

-- Table 4 goes about here --

The two analyses in this example would likely lead to the same conclusions -- the linear and linear × fumigant effects seem important ("significant"), but the quadratic and quadratic × fumigant effects are much less so. The calculations do differ, however, and in some settings the difference between an F-value of 1.47 and one of 1.88 (with 1 and 35 d.f.) might matter to someone.

We find this second example especially instructive because it serves to remind us that "nice" single degree of freedom contrasts and interactions in an ANOVA may become nontrivial to deal with if we wish to adjust for a covariate.

3. Discussion

In the previous two sections we have noted that the controversy surrounding the calculation of SS's in unbalanced Model I ANOVA should carry over to Model I ANCOVA, even in the balanced case. Yet among those textbooks that we have examined (e.g., Bliss, 1970; Rao, 1973; Steel and Torrie, 1960) which include a presentation of analysis of covariance beyond the simple one-way case, we have found only examples of the non-hierarchical approach which use the formula for SS_{adj} given in expression (1).

We find the dominance of this non-hierarchical approach to ANCOVA in textbooks puzzling, because many of these books adopt (or at least discuss) hierarchical analyses in Model I ANOVA. There is a sense in which this dominance should be surprising because the ANCOVA model is itself the product of "hierarchical thinking". The basic ANCOVA model includes only a single regression coefficient for the covariate, and this assumes the absence of treatment × covariate interactions. Indeed, it is via the simple 1 d.f. adjustment that ANCOVA gains its strength. A completely consistent non-hierarchical approach to ANCOVA would adjust for treatment × covariate interactions as well. This would differ from both the "text-book" and the "hierarchical" approaches described here.

How much of a difference might we expect to find between the "textbook" and "hierarchical" approaches? In those examples which we have examined the difference between them is usually not great. We have found examples (e.g., Bliss, 1970, pp. 510ff) in which the analysis differ drastically, but they have exhibited substantial interaction, in which case the hierarchical main effect test is meaningless. The two examples in Section 2 are as dramatic as any others we have encountered where interaction is not present. Yet, such differences are of the same order of magnitude as those between the "textbook" F-values, and those calculated by doing an ANOVA of $y - \hat{\gamma}x$ (see Cochran and Cox, 1957; p. 87). Table 5 lists the 3 sets of F-values for their example analyzed here in Table 4. If we go to the trouble of doing the "correct" analysis, instead of doing an ANOVA on $y - \hat{\gamma}x$, then perhaps we should worry a little about whether we should do the "textbook" or the "hierarchical" analysis.

Although we have not carefully examined the standard computing packages to see how they handle this problem, it is our impression that most of the detailed ANCOVA programs for balanced layouts use the "textbook" formula, while the more general ANCOVA programs do a regression-like approach. In the latter case ANCOVA output will be of the textbook (non-hierarchical) or hierarchical form depending on which approach the package uses for its regression ANOVA table. Since there are some programs that report both hierarchical and non-hierarchical SS's, they have the ability to produce both types of ANCOVA calculations.

-- Table 5 goes about here --

Our purpose here has not been to advocate the hierarchical approach to ANCOVA calculations, although it is the one we favor. Rather we have tried to point out what we believe to be an anomaly in the standard textbook treatment of ANCOVA that disguises the fact that there *should be* some controversy over what is the correct method in the Model I ANCOVA, even in the balanced case. Thus, those who oppose the non-hierarchical approach in the case of unbalanced ANOVA, should also eschew it in the analysis of covariance.

Table 1.

(a) Table 1 from Cochran (1957) Showing Sums of Squares and Products for ANCOVA

	D. f.	(x^2)	(xy)	(y^2)
Treatments	$(t-1)$	T_{xx}	T_{xy}	T_{yy}
Error	f_e	E_{xx}	E_{xy}	E_{yy}
Sum	$t-1+f_e$	S_{xx}	S_{xy}	S_{yy}

(b) Table 2 from Cochran (1957) Showing Partition of E_{xx} and S_{xx} Into Components

	(y^2)	Regression D. f.	Regression S. s.	Deviations D. f.	Deviations S. s.	Deviations M. s.
Error	E_{yy}	1	E_{xy}^2/E_{xx}	f_e-1	$E_{yy}-E_{xy}^2/E_{xx}$	s_e^2
Sum	S_{yy}	1	S_{xy}^2/S_{xx}	$t+f_e-2$	$S_{yy}-S_{xy}^2/S_{xx}$	
Treatments (by subtraction)				$t-1$	$T_{yy}-S_{xy}^2/S_{xx}+E_{xy}^2/E_{xx}$	s_t^2

Table 2. Example from Federer (1957; Table 9, p. 356)

NUMBER OF SALABLE FLOWERS (ROSES), Y_{ijk}, OPENED IN ONE MONTH (APRIL) AND LOCATION ON GREENHOUSE BENCH, X_{ijk}—7/15 PLANTING.

Treatment	Rep. I					Rep. II					Total		
	n_{i1}	Y	X	Y	X	n_{i2}	Y	X	Y	X	$n_{i.}$	$Y_{i..}$	$X_{i..}$
1	1	102	15	—	—	2	71	10	79	11	3	252	36
2	2	84	9	81	7	1	76	14	—	—	3	241	30
3	2	67	5	83	4	1	74	2	—	—	3	224	11
4	1	71	11	—	—	2	51	4	63	5	3	185	20
5	1	53	2	—	—	2	63	8	61	7	3	177	17
Total	7	$Y_{.1.}=541$; $X_{.1.}=53$				8	$Y_{.2.}=538$; $X_{.2.}=61$				15	1079	114

Table 3. Analyses of Data from Table 2

Source of Variation	df	Sum of Squares and Products (x^2)	(xy)	(y^2)	Textbook Analysis SS_{adj}	F	Hierarchical Analysis SS_{adj}	F
Blocks (B)	1	0.01	-2.01	376.00	*		*	
Treatment Eliminating Blocks (T|B)	4	136.12	303.81	1320.96	-		667.91	2.91
Treatment Eliminating Blocks and Inter-Action (T|B,B×T)	4	172.53	367.33	1606.67	826.14	3.60	-	
Interaction Eliminating Blocks and Treatment (B×T|B,T)	4	89.47	139.80	491.47	273.03	1.19	273.03	1.19
Within	5	4.00	6.00	238.50	229.50			
Total	14	229.60	447.60	2426.93				

*Federer computes no covariance adjusted block SS. The "textbook" analysis yields SS(B|T,B×T,x) = 156.74, while the hierarchical analysis yields SS(B|T,x) = 175.47.

Table 4. Analysis of Covariance for Data in Cochran and Cox (1957), Table 3.1, Including Values Reported in Tables 3.9 and 3.12

Source of Variation	df	Sum of Squares and Products (x^2)	(xy)	(y^2)	Textbook Analysis SS_{adj}	F	Hierarchical Analysis SS_{adj}	F
Blocks	3	159,618	175,873	289,427	110,057*		110.057*	
Treatments	8							
Linear	1	3,081	-13,276	57,207	103,465	14.51	95,321	13.37
Quadratic	1	2,204	8,285	31,140	10,474	1.47	13,405	1.88
Linear × Fumigant	3	22,975	-6,837	43,408	107,966	5.05	108,768	5.08
Quadratic × Fumigant	3	882	2,606	25,693	19,699	0.92	19,699	0.92
Error	36	121,408	189,278	544,690	249,601			
Total	47	310,168	355,929	991,565				

*This is SS(blocks|treatment,x) in both cases.

Table 5. F-Values for Components in Table 4

Source of Variation	df	F-Values: ANOVA on $y - \hat{\gamma}x$	Textbook	Hierarchical
Linear	1	14.88	14.51	13.37
Quadratic	1	1.50	1.47	1.88
Differences in Linear	3	5.64	5.05	5.08
Differences in Quadratic	3	0.92	0.92	0.92
Error	35			

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

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1. REPORT NUMBER: Technical Report #204	2. GOVT ACCESSION NO.: AD-A102 617
3. RECIPIENT'S CATALOG NUMBER	
4. TITLE (and Subtitle): TEXTBOOK ANALYSIS OF COVARIANCE -- IS IT CORRECT?	5. TYPE OF REPORT & PERIOD COVERED: To May, 1981
	6. PERFORMING ORG. REPORT NUMBER: TR #204
7. AUTHOR(s): Christopher Bingham, Stephen E. Fienberg	8. CONTRACT OR GRANT NUMBER(s): N00014-80-C-0637
9. PERFORMING ORGANIZATION NAME AND ADDRESS: Department of Statistics, Carnegie-Mellon University, Pittsburgh, PA 15213	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS: Contracts Office, Carnegie-Mellon University, Pittsburgh, PA 15213	12. REPORT DATE
	13. NUMBER OF PAGES: 18
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)	15. SECURITY CLASS. (of this report): Unclassified
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

APPROVED FOR PUBLIC RELEASE: DISTRIBUTION UNLIMITED

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Hierarchical analysis, covariance

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

For unbalanced Model I ANOVA data there is a controversy in the literature -- should the partial SS's be computed hierarchically or should they be computed after fitting all other model terms including higher-order interactions? A similar problem exists in the Model I analysis of covariance (ANCOVA), yet the usual prescription given in textbooks uses only the non-hierarchical approach. Using two classic examples of ANCOVA, this paper examines the two approaches to ANCOVA calculations.